Math 131 - Fall 2023 - Common Final Exam, version A Solutions

1. Consider the function $f(x) = \dfrac{8x^2 - 35}{1 - 10x^2}$.

   (a) (3 points) Evaluate $\lim_{x\to\infty} f(x)$.

Solution:	1 pt	some correct attempt at evaluating the limit
	2 pt	correct work and solution

   (a) $-\frac{4}{5}$

   (b) (3 points) Evaluate $\lim_{x\to 0} f(x)$.

Solution:	1 pt	some correct attempt at evaluating the limit
	2 pt	correct work and solution

   (b) -35

2. The following table gives some values for $s(t)$, the position of an object (in km) after t minutes.

t	3	4	5	6	7
$s(t)$	24.8	42.0	43.5	46.2	47.0

   (a) (3 points) Find the average velocity of the object from $t = 4$ to $t = 7$.

Solution:	1 pt	some difference quotient is evaluated
	1 pt	correct dq and work
	1 pt	units

   The average velocity (with units) is $\frac{5}{3}$ **km per min**

   (b) (3 points) Estimate the instantaneous velocity at $t = 6$. Show some work that supports your answer.

Solution:	1 pt	some difference quotient is evaluated with an endpoint at t=6
	1 pt	correct dq and work
	1 pt	units

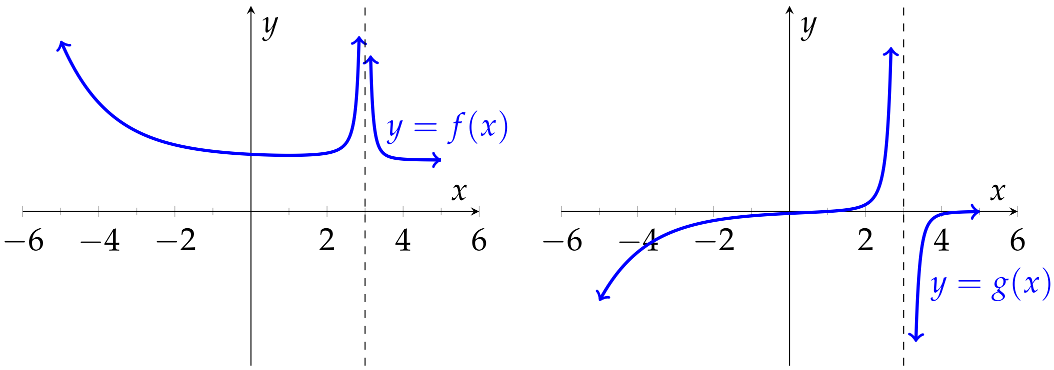
   Reasonable values:

   $$\frac{s(7) - s(6)}{7 - 6} = 0.8 \text{ or } \frac{s(6) - s(5)}{6 - 5} = 2.7,$$

   or the average of these two which is 1.75.

   According to the work above, the instantaneous velocity (with units) is 0.8 **or** 2.7 **or** 1.75 **km per min**.

3. The graphs below show functions f on the left and g on the right[0.8 or 2.7 or 1.75 km per min].



(a) (2 points) Choose the option below which best describes the relationship between these graphs.

Solution: 2 pt for correct choice

- ◯ f is the derivative of g.
- √ **g is the derivative of f.**
- ◯ Neither function is the derivative of the other.

(b) (3 points) Support your choice above with a complete sentence which includes at least one fact about slope or concavity at a point.

Solution:

2 pt	correct statement about slope or concavity which supports answer
1 pt	correct statment which rules out the other

Examples of reasons supporting $f' = g$:

- $f'(1) \approx 0$ and $g(1) = 0$
- f is everywhere concave up and g is everywhere increasing

$g' \neq f$ because $g'(1) \approx 0$ but $f(1) > 0$.

4. Consider the function $f(t) = \cos\big(\ln(t)\big)$.

(a) (3 points) Write the definition of f' as a limit involving h. (A formula for the derivative using shortcut rules is not worth credit here. You must use the definition of the derivative.)

Solution:

1 pt	some difference quotient with a limit
2 pt	correct dq with this function

$$f'(t) = \lim_{h \to 0} \frac{\cos\big(\ln(t+h)\big) - \cos\big(\ln(t)\big)}{h}$$

(b) (3 points) Estimate $f'(3)$ by evaluating the formula in your limit at an appropriate value of h.

Solution:

2 pt	evidence of plugging in a small value of h to the dq in (a)
1 pt	correct evaluation and answer

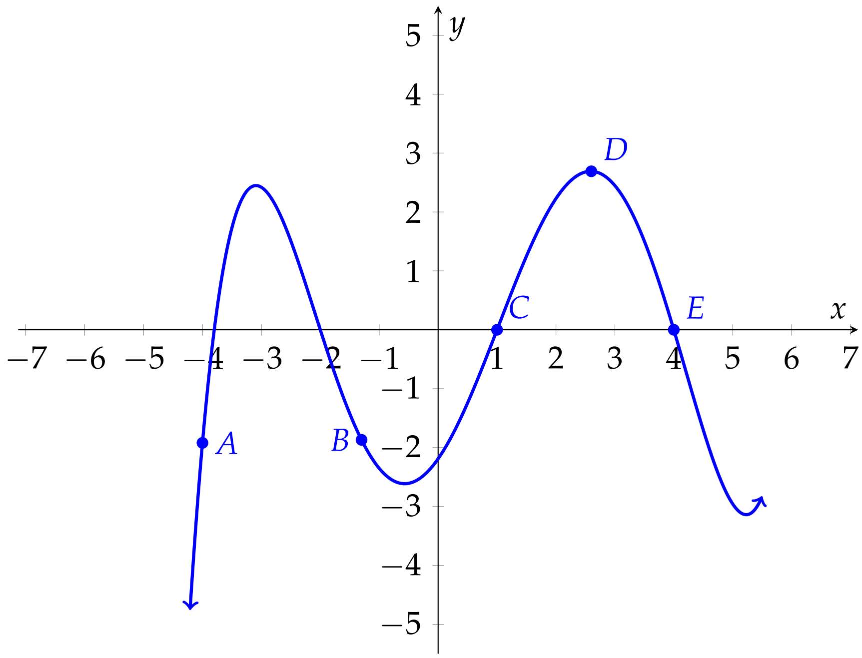
With $DQ(h) = \frac{\cos\left(\ln(3+h)\right) - \cos\left(\ln(3)\right)}{h}$:

h	-0.1	-0.01	-0.001	0.001	0.01	0.1
$DQ(h)$	-0.2992	-0.2971	-0.2969	-0.2968	-0.2966	-0.2944

Unfortunately if a student treats the cosine argument as degrees for the computation, all the above values are approximately -0.0001.

Based on the computations above, $f'(3) \approx$ -0.29 **or** -0.3

5. Use the graph of $p(x)$ below to answer the following questions. Your answers should list the letters of all labeled points which apply, or "NA" if no labeled points apply.



Solution: Parts a,b: +1 for each correct and -1 for each incorrect response, with a minimum of 0 part c: 2 pt for correct response, -1 for each incorrect additional response, with a minimum of 0

(a) (2 points) At which point(s) is $p'(x)$ negative?

(a) *B, E*

(b) (2 points) At which point(s) is $p''(x)$ approximately zero?

(b) *C, E*

(c) (2 points) At which points is $p''(x)$ positive?

(c) *B*

6. The quantity of a drug in a patient's bloodstream (in mg) t minutes after an injection is $C(t)$.

(a) (3 points) Give the practical meaning of $C(3) = 250$ in a sentence with correct units.

Solution:		
	1 pt	for correct meaning of input
	1 pt	correct meaning of output
	1 pt	correct units throughout

"3 minutes after the injection, the patient's bloodstream contains 250 mg of the drug."

(b) (3 points) Give the practical meaning of $C'(3) = -20$ in a sentence with correct units.

Solution:

1 pt	for correct meaning of input
1 pt	correct meaning of output
1 pt	correct units throughout

"3 minutes after the injection, the quantity of the drug in the bloodstream is decreasing at a(n instantaneous) rate of 20 mg per minute."

7. (6 points) Find $g''(1)$ if $g(x) = 2\sqrt{x} + x^8 - (5x+1)^2$. Show all your steps.

Solution:	2 pt	power rule on 1st two terms
	2 pt	chain rule or FOILing
	2 pt	finding second derivative correctly, evaluation and answer

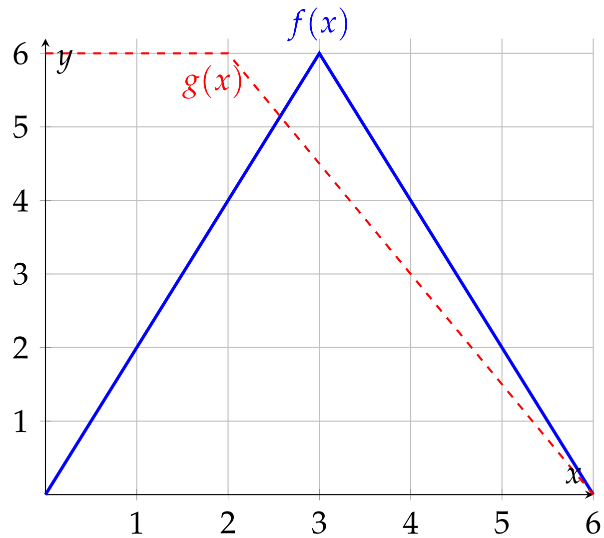
$g''(1) =$ ______ $5\frac{1}{2}$ ______.

8. (6 points) Find the equation of the tangent line to the graph of $w(x) = \dfrac{5x+3}{x-1}$ at $x = 0$. Give your answer in slope-intercept form.

Solution:	3 pt	$w'(0)$
	1 pt	$w(0)$
	2 pt	putting it together

The tangent line is $y =$ $-8x$ - 3 . ______

9. (6 points) If $r(x) = f(x) \cdot g(x)$, use the graphs of f and g below to evaluate the following derivatives.



Solution:

2 pt	evidence of product rule for r′
2 pt	evaluation of r′(1)
2 pt	evaluation of r′(4)

$r'(x) = f'(x)g(x) + f(x)g'(x)$

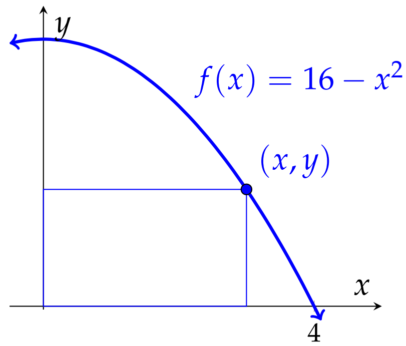
(a) Evaluate $r'(1)$

$r'(1) =$ ___12___

(b) Evaluate $r'(4)$.

$r'(4) =$ ___-12___

10. (6 points) Below is a rectangle inscribed under the graph of f.



Find the value of x on the interval $[0, 4]$ which gives the largest area of this rectangle. Your answer must make clear

- The function of x which you are maximizing,
- the domain of this function, and
- show that your answer gives a maximum value for the area.

Solution:		
	1 pt	evidence of some area formula
	1 pt	objective function of one variable to optimize
	1 pt	first derivative of objetive
	1 pt	find critical points of objective function
	1 pt	testing critical points and endpoints or at least clear statement of domain
	1 pt	correct answer

We want to maximize $A = xy$. Since $y = 16 - x^2$, we are maximizing the function $A = f(x) = x(16 - x^2) = 16x - x^3$ over the given interval $0 \le x \le 4$. The derivative of this function is $\frac{dA}{dx} = 16 - 3x^2$, and there are critical points at $x = \pm\frac{4}{\sqrt{3}}$. Only the positive critical point is in the interval $0 \le x \le 4$.

$$f(0) = 0$$

$$f\left(\frac{4}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}} \times \frac{32}{3} = \frac{128}{3\sqrt{3}} \qquad \text{Maximum value}$$

$$f(4) = 0$$

Therefore the area of the rectangle is maximized when $x = \frac{4}{\sqrt{3}}$.

According to the work above, the maximum area occurs at $x =$ $\frac{4}{\sqrt{3}} \approx 2.31$

11. Christopher runs a custom computer design business. He calculates his monthly revenue using the function

$$R(q) = 1800 \ln(20q + 40)$$

and his monthly costs using the function

$$C(q) = 4000 + 40q.$$

(a) (4 points) Find the quantity of computers q he needs to sell to maximize his profit.

Solution:	2 pt	correct derivative
	2 pt	for critical points ($\pi' = 0$ or MR = MC)

He will maximize his profit by making $q =$ **43** computers.

(b) (2 points) What is the profit he expects to make by selling the quantity of computers determined in part (a). Round your answer to the nearest dollar.

Solution:	1 pt	correct profit function either from (a) or newly written
	1 pt	correct evaluation

The maximum profit is **$6524**].

12. (4 points) Let $F(x)$ be an antiderivative of $f(x)$. If $\int_{-3}^{1} f(x)\,dx = 4$ and $F(1) = 5$, find $F(-3)$.

Solution:	2 pt	evidence of FTC
	2 pt	correct answer

$F(-3) =$ **1** .

13. (5 points) Find the indefinite integral

$$\int \left(4x^4 - \frac{4}{x} + \frac{9}{x^6}\right) dx.$$

Use C for the constant of integration. Show all steps.

Solution:	1 pt	for $4x^4$
	1 pt	for $\frac{4}{x}$
	2 pt	for $\frac{9}{x^6}$, correct exonent rules and antiderivative
	1 pt	constant of integration

The indefinite integral is $\frac{4}{5}x^5 - 4\ln|x| - \frac{9}{5}x^{-5} + C$.

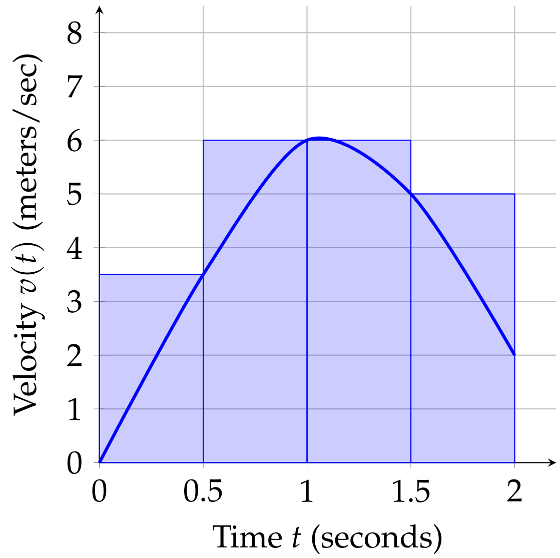
14. (6 points) Find the antiderivative $G(x)$ of $g(x)$ with $G(0) = 2$, where

$$g(x) = 2e^x - \sin(x) + 6x + 4.$$

Solution:	1 pt	each for antiderivatives of 4 terms
	1 pt	for attempt to solve for C
	1 pt	correct value of C

$G(x) = 2e^x + \cos(x) + 3x^2 + 4x - 1$

15. The figure shows the graph of the velocity (in meters per second) of a particle for $0 \leq t \leq 2$ and the rectangles used to estimate the distance traveled by a Riemann sum.



Solution: for parts a-d: 2 pt for correct solution part e:

1 pt	computation
1 pt	units

(a) (1 point) The rectangles represent

- ◯ a left Riemann sum.
- ◯ a right Riemann sum.
- √ **some other kind of Riemann sum.**

(b) (1 point) The estimate for distance traveled based on the indicated Riemann sum is

- √ **an overestimate.**
- ◯ an underestimate.
- ◯ either an underestimate or an overestimate; it is not possible to tell.

(c) (1 point) What is the value of n for the Riemann sum?

(c) **4**

(d) (1 point) What is the value of Δt for the Riemann sum?

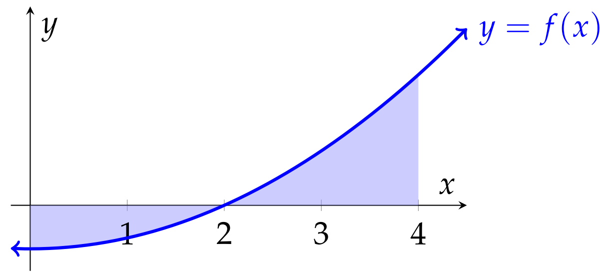
(d) **0.5**

(e) (2 points) Use the rectangles to estimate the total distance traveled by the particle for $0 \leq t \leq 2$.

Solution:

The distance traveled (with units) is approximately **10.25 meters**.

16. (5 points) Below is the graph of $y = f(x)$.



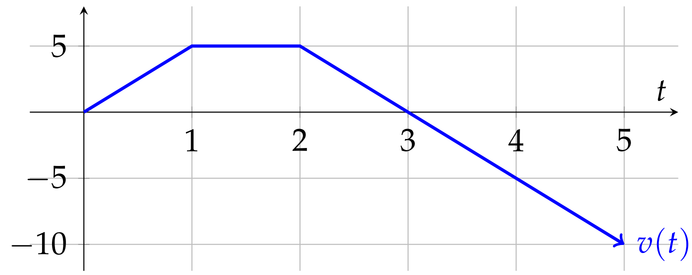
Write an expression with an integral or integrals which gives the area of the shaded region. Do not evaluate your expression.

Solution:

2 pt	some integral
2 pt	split up integral at $x = 2$
2 pt	change sign of $\int_0^2$

$$-\int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx$$

17. (5 points) The graph below gives the velocity of an object t seconds after it begins moving along a line. Use the graph to answer the following questions.



Solution:

2 pt	correct choice in (a)
2 pt	answer references area representing distance (this could be computation of distances
1 pt	consistent and supports (a)

(a) Is the object farther from its starting point at $t = 1$ or $t = 4$? Choose the best answer below.

◯ The object is farther from its starting point at $t = 1$.

√ **The object is farther from its starting point at** $t = 4$.

◯ The object is the same distance from its starting point at $t = 1$ and at $t = 4$.

(b) Use some information from the graph to explain in a sentence why your choice is correct.

Solution: "The postive area representing the distance from the start from $t = 0$ to $t = 3$ is greater than the negative area representing distance traveled back toward the start from $t = 3$ to $t = 4$."

Students may also show explicit computation of the net distance at $t = 3$ and $t = 4$.